On equations for neutrino propagation in matter

Paul M. Fishbane*
Physics Dept. and Institute for Nuclear and Particle Physics,
Univ. of Virginia, Charlottesville, VA 22903

Stephen G. Gasiorowicz**
Physics Dept. and Institute of Theoretical Physics,
University of Minnesota. Minneapolis, MN 55455

We study the dynamical equations for two-family neutrino oscillations in a medium of continuously-varying density. We can find explicit solutions to these equations in terms of series of nested integrals. These solutions can serve as a basis for numerical calculation of these processes or for further study of their analytical properties.

* email address pmf2r@virginia.edu
** email address gasior@umn.edu

I. Introduction

The presence of neutrino oscillations [1-2] has renewed interest in the question of oscillations within matter [3-5]. The early work ("MSW") of Refs. 3 and 4 solved the problem of propagation within a medium of constant density, and it is possible to treat nonconstant density by numerical means. It is nevertheless always useful to think about an analytic approach [6] in order to develop insight and understanding. Since the case we study here is that of a two-channel problem, some aspects of the methods we describe are also applicable to spins in varying magnetic fields and other two-channel order-dependent problems.

For convenience we recall here the MSW results [3, 4], constant density We assume a two-channel approximation to neutrino mixing and give the amplitude $T(t)$ for a neutrino beam of energy E passing through a medium of constant electron density N_e given some initial neutrino flavor mixture $\psi(0)$, namely

$$\psi(t) = T(t)\psi(0). \tag{1.1}$$

(The time t can be interchangably viewed as the thickness x of the medium.) Here the two-vector weak state (i.e., flavor basis) is $\psi(t) = \begin{bmatrix} \nu_e \\ \nu_\mu \end{bmatrix}$. The transition amplitude T is

$$T = \cos\phi' + i\sin\phi'\cos(2\theta')\sigma_z - i\sin\phi'\sin(2\theta')\sigma_x \tag{1.2}$$

where the primed variables contain the effect of the matter:

$$\begin{aligned} \delta m^2 &\to \delta m'^2 = \delta m^2\sqrt{\left(\cos 2\theta - \frac{2EA}{\delta m^2}\right)^2 + \sin^2 2\theta} \\ \varphi &\to \varphi' = \frac{\delta m'^2}{4E}t \\ \theta &\to \theta' \ni \sin^2 2\theta' = \frac{\sin^2 2\theta}{\left(\cos 2\theta - \frac{2EA}{\delta m^2}\right)^2 + \sin^2 2\theta} \end{aligned} \tag{1.3}$$

with

$$A = \sqrt{2}G_F N_e. \tag{1.4}$$

The mass parameter $\delta m^2 = m_2^2 - m_1^2$, where m_i^2 is the i^{th} mass eigenvalue, is positive. We recover the vacuum result, Cabibbo angle θ, for $A = 0$.

This introduction sets some notation and recalls well-known results. Below we shall be concerned with variable density.

II. Passage through a medium of variable density

The Schrödinger equation for a two-family weak state $\psi(t)$ propagating through a medium of electron density N_e is

$$i\frac{d\psi}{dt} = (H_F + W)\psi, \tag{2.1}$$

where, using the approximation $m_i << E$ and after pieces in the hamiltonian proportional to the unit matrix are removed—they lead only to overall phases, as we describe further below—we have

$$H_F = \frac{\delta m^2}{4E}\begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \tag{2.2}$$

and

$$W = \begin{bmatrix} \sqrt{2}G_F N_e & 0 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \tag{2.3}$$

Here H_F is related to the mass eigenstates by the Cabibbo matrix $V = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$,

$$H_F = V\left\{\frac{1}{2E}\begin{bmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{bmatrix}\right\}V^\dagger \equiv VH_D V^\dagger \,. \tag{2.4}$$

A solution to Eq. (2.1) will give us the amplitude $T(t)$, through Eq. (1.1).

It is convenient to approach the dynamical equation (2.1) by starting in the mass basis, using the transformed states $\phi \equiv V^\dagger\psi$. We do so by multiplying Eq. (2.1) by $V^\dagger$, giving

$$i\frac{d\phi}{dt} = \left(H_D + V^\dagger W V\right)\phi \,. \tag{2.5}$$

Next we remove the factor H_D by defining a new function ξ by

$$\phi \equiv e^{-iH_D t}\xi \,.$$

Starting from Eq. (2.5) it is easy to see that this function obeys the equation

$$i\frac{d\xi}{dt} = e^{iH_D t}V^\dagger W V e^{-iH_D t}\xi \,. \tag{2.6}$$

In this expression, only the mass difference enters into the exponentials. To see this, define

$$\bar{E} \equiv E + \frac{m_1^2 + m_2^2}{4E} \quad \text{and} \quad \Delta \equiv -\frac{\delta m^2}{4E} \,. \tag{2.7}$$

In terms of these quantities, H_D reads

$$H_D = \bar{E} + \sigma_3\Delta$$

The portion of H_D proportional to the unit matrix commutes through the quantities in Eq. (2.6) and cancels, leaving

$$i\frac{d\xi}{dt} = e^{i\Delta\sigma_3 t}V^\dagger W V e^{-i\Delta\sigma_3 t}\xi \,. \tag{2.8}$$

We next write out in 2×2 form the quantity multiplying ξ on the right side of Eq. (2.8), using in particular the identity

$$e^{ia\sigma_3} = \cos a + i\sigma_3 \sin a = \begin{bmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{bmatrix}. \tag{2.9}$$

We find

$$
\begin{aligned}
e^{i\Delta\sigma_3 t}V^\dagger W V e^{-i\Delta\sigma_3 t} &= \frac{1}{2}A\begin{bmatrix} 1+\cos 2\theta & e^{2i\Delta t}\sin 2\theta \\ e^{-2i\Delta t}\sin 2\theta & 1-\cos 2\theta \end{bmatrix} \\
&\to \frac{1}{2}A\begin{bmatrix} \cos 2\theta & e^{2i\Delta t}\sin 2\theta \\ e^{-2i\Delta t}\sin 2\theta & -\cos 2\theta \end{bmatrix}
\end{aligned} \tag{2.10}
$$

In the last step we have taken out the piece proportional to the unit matrix; again, it contributes only an overall phase to ξ. (This can be seen in a variety of ways. For example, one can define a new function $\eta \equiv \exp\left(+\frac{i}{2}\int_0^t A(t')\,dt'\right)\xi$ and show that the equation for η is identical to that for ξ but with the last form in Eq. (2.10) on the right multiplying η.) We can finally transform away the term in Eq. (2.10) proportional to σ_3. We define the new function ζ by

$$
\zeta \equiv \exp\left(+\frac{i}{2}\sigma_3 I(t)\cos 2\theta\right)\xi = \begin{bmatrix} \exp\left(\frac{i}{2}I(t)\cos 2\theta\right) & 0 \\ 0 & \exp\left(-\frac{i}{2}I(t)\cos 2\theta\right) \end{bmatrix}\xi \tag{2.11}
$$

where

$$
I(t) \equiv \int_0^t A(t')\,dt'. \tag{2.12}
$$

The function ζ obeys the simpler equation

$$
\begin{aligned}
i\frac{d\zeta}{dt} &= e^{\frac{i}{2}\sigma_3 I\cos 2\theta}\begin{bmatrix} 0 & \frac{1}{2}Ae^{2i\Delta t}\sin 2\theta \\ \frac{1}{2}Ae^{-2i\Delta t}\sin 2\theta & 0 \end{bmatrix} e^{-\frac{i}{2}\sigma_3 I\cos 2\theta}\zeta \\
&= \begin{bmatrix} 0 & \frac{1}{2}Ae^{i(2\Delta t + I\cos 2\theta)}\sin 2\theta \\ \frac{1}{2}Ae^{-i(2\Delta t + I\cos 2\theta)}\sin 2\theta & 0 \end{bmatrix}\zeta \\
&\equiv P(t)\zeta.
\end{aligned} \tag{2.13}
$$

The last line defines the matrix $P(t)$, in terms of which we can write our formal solution to this equation. We remark in particular, for later use, that the 21 element P_{21} equals the complex conjugate of the 12 element P_{12}. We will below express the solution to Eq. (2.13) in terms of P_{12} and P_{12}^*.

The form that $P(t)$ takes is easy to understand. The central matrix in the first line of Eq. (2.13) is a linear combination of σ_1 and σ_2, while the external factors $e^{\pm\frac{i}{2}\sigma_3 I\cos 2\theta}$ take the form of a rotation about the 3-axis. Their effect is then to rotate the combination of σ_1 and σ_2 to give a different combination that lies at a different angle—that takes the form of the original combination but with the phase shifted. This is indeed what happens, as the explicit calculations that give us $P(t)$ show.

Solution of the equation for ζ. It is clear from the matrix structure of Eq. (2.13) that the commutator $[P(t), P(t')] \neq 0$, so that the solution to Eq. (2.13) involves an ordering. Formally, the solution is

$$\zeta(t) = \mathbf{T}\left[\exp\left(-i\int_0^t P(s)\,ds\right)\right]\zeta(0) \equiv M(t)\zeta(0), \tag{2.14a}$$

where the time ordering operator T ensures that the matrices P in the expression are ordered so that P with a later argument stands to the left of P with an earlier argument. We may write

$$M(t) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_0^t ds_1\int_0^t ds_2\cdots\int_0^t ds_n \mathbf{T}\left[P(s_1)P(s_2)\cdots P(s_n)\right],$$

and it is a standard exercise to show that this is equivalent to

$$M(t) = \sum_{n=0}^{\infty}(-i)^n\int_0^t P(s_1)\,ds_1\int_o^{s_1} P(s_2)\,ds_2\cdots\int_0^{s_{n-1}} P(s_n)\,ds_n \tag{2.14b}$$

Finally we can separate out the explicit matrix elements of M by using the matrix structure of P, namely $P(t) = P_x(t)\sigma_1 + P_y(t)\sigma_2$. This leads immediately to

$$M_{11} = 1 - \int_0^t P_{12}(t')\left(\int_0^{t'} P_{12}^*(t'')\,dt''\right)dt' + \ldots \tag{2.15a}$$

$$M_{12} = -i\int_0^t P_{12}(t')\,dt' + i\int_0^t P_{12}(t')\left(\int_0^{t'} P_{12}^*(t'')\left(\int_0^{t''} P_{12}(t''')\,dt'''\right)dt''\right)dt' - \ldots \tag{2.15b}$$

$$M_{21} = M_{12}\big|_{P_{12}\leftrightarrow P_{12}^*} \tag{2.15c}$$

$$M_{22} = M_{11}^* \tag{2.15d}$$

Alternatively, direct differentiation of the group of Eqs. (2.15) shows that it satisfies the necessary condition $\frac{dM}{dt} = -iPM$.

The solution given by Eqs. (2.15) is essentially a power series in A. The order dependence of the result is contained in the fact that the integrals in M are nested. We can write the transition amplitude T in terms of M by undoing our series of transformations, leaving

$$T(t) = V\begin{bmatrix} e^{-i\left(\Delta t + \frac{1}{2}I(t)\cos 2\theta\right)} & 0 \\ 0 & e^{i\left(\Delta t + \frac{1}{2}I(t)\cos 2\theta\right)} \end{bmatrix} M(t)V^{\dagger} \tag{2.16}$$

Generally speaking the nested integrals in M are complicated, even for the case of constant density. (Of course numerical integration is always possible.)

III. Second order equation

Equation (2.13) represents two coupled first order differential equations. These equations can be rewritten as a single uncoupled second order equation, and the second order

equation lends itself to a variety of treatments beyond the formal solution we have already expressed. In terms of the explicit components of ζ, Eq. (2.13) reads

$$\begin{bmatrix}\dot{\zeta}_1\\ \dot{\zeta}_2\end{bmatrix}=\begin{bmatrix}0 & \alpha\\ \beta & 0\end{bmatrix}\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}=\begin{bmatrix}\alpha\zeta_2\\ \beta\zeta_1\end{bmatrix}, \tag{3.1}$$

where

$$\alpha \equiv -iP_{12} \quad \text{and} \quad \beta \equiv -iP_{21} = -iP_{12}^*. \tag{3.2}$$

One more derivative of, say, the upper component gives $\ddot{\zeta}_1 = \dot{\alpha}\zeta_2 + \alpha\dot{\zeta}_2 = \dfrac{\dot{\alpha}}{\alpha}\dot{\zeta}_1 + \alpha\beta\zeta_1$, or

$$\ddot{\zeta}_1 - \frac{\dot{\alpha}}{\alpha}\dot{\zeta}_1 - \alpha\beta\zeta_1 = 0 \tag{3.3}$$

Using the explicit expressions for α and β, we find that

$$\frac{\dot{\alpha}}{\alpha} = \frac{\dot{A}}{A} + i\left(2\Delta + A\cos 2\theta\right) \quad \text{and} \quad \alpha\beta = -\left(\frac{1}{2}A\sin 2\theta\right)^2 .$$

Thus we have finally the second order equation

$$\ddot{\zeta}_1 - \frac{\dot{A}}{A}\dot{\zeta}_1 + i\left(2\Delta + A\cos 2\theta\right)\dot{\zeta}_1 + \left(\frac{1}{2}A\sin 2\theta\right)^2\zeta_1 = 0 \tag{3.4}$$

It is worthwhile noting that the quantity I, Eq. (2.12), does not appear in this equation. We may take the required two boundary conditions to be $\zeta_1(0)$ and $\zeta_2(0) = \dot{\zeta}_1(0)/\alpha(0)$. Once we find the solution for $\zeta_1(t)$, we have $\zeta_2(t) = \dfrac{1}{\alpha(t)}\dfrac{d\zeta_1(t)}{dt}$. We also remark here that we have verified that our formal solution to the equations for ζ, Eq. (2.15), satisfies Eq. (3.4).

Recovery of constant density case. For the MSW case (constant $A \equiv A_0$), reviewed in Section I, Eq. (3.4) takes the form

$$\ddot{\zeta}_1 - ib_0\dot{\zeta}_1 + c_0^2\zeta_1 = 0 \tag{3.5}$$

where

$$b_o = 2\Delta + A_0\cos 2\theta \quad \text{and} \quad \mathrm{c}_0 = \frac{1}{2}A_0\sin 2\theta\ . \tag{3.6}$$

If in addition we define

$$\omega \equiv \sqrt{b_0^2 + 4c_0^2}\ , \tag{3.7}$$

then the solution to this equation is

$$\zeta_1(t) = B_1\exp\left(\frac{it}{2}[\omega + b_0]\right) + C_1\exp\left(\frac{it}{2}[-\omega + b_0]\right) \tag{3.8a}$$

$$\zeta_2(t) = B_2\exp\left(\frac{it}{2}[\omega - b_0]\right) + C_2\exp\left(\frac{it}{2}[-\omega - b_0]\right), \tag{3.8b}$$

where

$$B_1 = \frac{1}{\omega}\left(\frac{1}{2}\zeta_1(0)[\omega - b_0] - c_0\zeta_2(0)\right)$$
$$C_1 = \frac{1}{\omega}\left(\frac{1}{2}\zeta_1(0)[\omega + b_0] + c_0\zeta_2(0)\right)$$
$$B_2 = \frac{1}{\omega}\left(\frac{1}{2}\zeta_2(0)[\omega + b_0] - c_0\zeta_1(0)\right)$$
$$C_2 = \frac{1}{\omega}\left(\frac{1}{2}\zeta_2(0)[\omega - b_0] + c_0\zeta_1(0)\right)$$

This solution allows us to identify $M(t)$ through Eq. (2.14) and hence the transition amplitude T through Eq. (2.16) adapted to constant A, namely

$$T(t) = V\begin{bmatrix} e^{-\frac{it}{2}(2\Delta + A_0\cos 2\theta)} & 0 \\ 0 & e^{\frac{it}{2}(2\Delta + A_0\cos 2\theta)} \end{bmatrix} MV^{\dagger} = V\begin{bmatrix} e^{-\frac{it}{2}b_0} & 0 \\ 0 & e^{\frac{it}{2}b_0} \end{bmatrix} MV^{\dagger} \qquad (3.9)$$

The result of the exercise is

$$T = \frac{1}{2\omega}V\begin{bmatrix} (\omega - b_0)e^{i\omega t/2} + (\omega + b_0)e^{-i\omega t/2} & -2c_0\left(e^{i\omega t/2} - e^{-i\omega t/2}\right) \\ -2c_0\left(e^{i\omega t/2} - e^{-i\omega t/2}\right) & (\omega + b_0)e^{i\omega t/2} + (\omega - b_0)e^{-i\omega t/2} \end{bmatrix} V^{\dagger} \quad (3.10)$$

Then specific calculation shows, for example,

$$T_{12}\left(= T_{e\mu}\right) = \frac{2i}{\omega}\Delta \sin 2\theta \sin(\omega t/2). \qquad (3.11)$$

This can be put into the canonical form of Ref. [4], $-i(\sin\varphi_m)(\sin 2\theta_m)$, where the subscript m indicates the propagation is in material, and where $\varphi_m = \frac{\delta m_{eff}^2}{4E}t$, if we identify

$$\sin 2\theta_m = \frac{2|\Delta|\sin 2\theta}{\omega} = \frac{\sin 2\theta}{\sqrt{1 - \frac{4EA}{\delta m^2}\cos 2\theta + \left(\frac{2EA}{\delta m^2}\right)^2}} = \frac{\sin 2\theta}{\sqrt{\left(\cos 2\theta - \frac{2EA}{\delta m^2}\right)^2 + \sin^2 2\theta}} \quad (3.12a)$$

and

$$\delta m_{eff}^2 = 2E\omega = \delta m^2\sqrt{1 - \frac{4EA}{\delta m^2}\cos 2\theta + \left(\frac{2EA}{\delta m^2}\right)^2} = \delta m^2\sqrt{\left(\cos 2\theta - \frac{2EA}{\delta m^2}\right)^2 + \sin^2 2\theta} \,. \quad (3.12b)$$

Indeed, in terms of these new variables the full transition amplitude is

$$T = \cos\varphi_m + \left(i\sin\varphi_m\cos 2\theta_m\right)\sigma_z - \left(i\sin\varphi_m\sin 2\theta_m\right)\sigma_x \,. \qquad (3.13)$$

This is the full canonical form described in Section I for propagation in a medium of constant density.

Adiabatic Expansion. If the factor in Eq. (3.4) that contains the derivative of A is small compared to the other factors, one can make a systematic adiabatic expansion [6zz] in terms of it about the 0th order (MSW) answer. To do so, it is useful to recast the solution technique somewhat. We shall first take the starting point of the neutrino beam, at $t = 0$, to specify the constant background level of the material density factor, i.e., $A(0) = A_0$. We

leave the boundary conditions $\zeta_1(0)$ and $\zeta_2(0)$ unspecified for the moment but remark that using Eq. (2.13) the boundary condition for $\zeta_2(0)$ gives us alternatively a condition for the derivative of ζ_1 at $t = 0$: $\frac{d\zeta_1}{dt}(0) = -iP_{12}(0)\zeta_2(0) = -\frac{i}{2}A_0(\sin 2\theta)\zeta_2(0)$.

We see from our earlier solution of the constant density case (Eq. 3.8) that the t-dependence is contained in a pair of phases. An alternative way to derive these phases in the constant density case is through a solution ansatz of the schematic form

$$\zeta_1 = R_0 \exp(iS_0(t)), \tag{3.14}$$

where R_0 is constant and where $S_0(t = 0) = 0$. The real and imaginary parts of Eq. (3.5) lead to the following equations for $S_0(t)$:

$$\begin{gathered}\left(\frac{dS_0}{dt}\right)^2 - b_0\frac{dS_0}{dt} - c_0^2 = 0\\ \frac{d^2S_0}{dt^2} = 0.\end{gathered} \tag{3.15}$$

Together with the vanishing of S_0 at $t = 0$, these imply that

$$S_0(t) = \lambda t \tag{3.16}$$

where

$$\lambda^2 - b_0\lambda - c_0^2 = 0 \tag{3.17}$$

The solution of the quadratic equation gives

$$\lambda_\pm = \frac{1}{2}(b_0 \pm \omega),\ \text{where } \omega \equiv \sqrt{b_0^2 + 4c_0^2}. \tag{3.18}$$

Thus, as indeed Eq. (3.8) shows, a better ansatz for the solution is

$$\zeta_1(t) = R_+e^{i\lambda_+t} + R_-e^{i\lambda_-t} \tag{3.19}$$

The boundary conditions for $\zeta_1(t)$ give us immediately $R_+ + R_- = \zeta_1(0)$ and $R_+\lambda_+ + R_-\lambda_- = -\frac{1}{2}A_0\zeta_2(0)\sin 2\theta$; in turn these last two relations determine

$$R_\pm = \frac{\zeta_1(0)}{2} \mp \frac{b_0\zeta_1(0) + A_0\zeta_2(0)\sin 2\theta}{2\omega}. \tag{3.20}$$

Having reviewed the 0th order (constant density) problem, we go on to include time (distance) dependence in the material density. We accordingly write the input density as

$$A(t) = A_0(1 + f_1(t)), \tag{3.21}$$

where $f_1(t) \ll 1$ for all t in the problem and $f_1(t \le 0) = 0$. We extend our ansatz for the solution to the form

$$\zeta_1(t) = R_+(t)\beta_+e^{iS_+(t)} + R_-(t)\beta_-e^{iS_-(t)}, \tag{3.22a}$$

where

$$R_\pm(t) = R_\pm\left(1 + \rho_1^\pm(t)\right) \tag{3.22b}$$

$$S_\pm(t) = \lambda_\pm t + \sigma_1^\pm(t) \tag{3.22c}$$

The quantities with subscript "1" are all small; moreover, $\rho_1^\pm(0) = \sigma_1^\pm(0) = 0$. We also set

$$b(t) = b_0 + b_1(t),\ \text{where } b_1(t) = A_0 f_1(t)\cos 2\theta \tag{3.23a}$$

$$c(t) = c_0 + c_1(t),\ \text{where } c_1(t) = (A_0 f_1(t)\sin 2\theta)/2. \tag{3.23b}$$

We now insert our ansatz into Eq. (3.5). The spirit of the adiabatic expansion is to keep only first order terms in quantities with the subscript "1." In addition, we insist that the coefficients of $\exp(iS_\pm(t))$ vanish separately.

The real and imaginary parts of the coefficients of $\exp(iS_\pm)$ give respectively

$$\frac{d^2\rho_1^\pm}{dt^2} \mp \omega \frac{d\sigma_1^\pm}{dt} + K_\pm f_1 = 0 \tag{3.24a}$$

$$\frac{d^2\sigma_1^\pm}{dt^2} \pm \omega \frac{d\rho_1^\pm}{dt} - \lambda_\pm \frac{df_1}{dt} = 0, \tag{3.24b}$$

where

$$K_\pm \equiv A_0(\lambda_\pm \cos 2\theta + c_0 \sin 2\theta) = 2\lambda_\pm(-2\Delta \pm \omega)/2. \tag{3.25}$$

Equations (3.24) contain only derivatives of the functions we seek, so they are in fact two coupled first order equations for the functions

$$v_1^\pm \equiv \frac{d\sigma_1^\pm}{dt} \quad \text{and} \quad u_1^\pm \equiv \frac{d\rho_1^\pm}{dt}. \tag{3.26}$$

To the equations for $v_1^\pm$ and $u_1^\pm$ we add boundary conditions that follow from $d\zeta_1/dt = 0$, namely $v_1^\pm(0) = 0 = u_1^\pm(0)$. As we shall see, these boundary conditions guarantee that σ_1 and ρ_1 remain small (i.e., $O(f_1)$).

We decouple the two equations for $v_1^\pm$ and $u_1^\pm$ by taking one more derivative of, say, Eq. (3.24a), giving a single second order equation for $u_1^\pm$:

$$\frac{d^2u_1^\pm}{dt^2} + \omega^2 u_1^\pm = 2\Delta\lambda_\pm \frac{df_1}{dt}, \tag{3.27}$$

where for the coefficient of df_1/dt we have used $\pm\omega\lambda_\pm - K_\pm = 2\Delta$. This equation is solved by a standard Green's function $G(t-t')$ that satisfies

$$\frac{d^2G(t-t')}{dt^2} + \omega^2 G(t-t') = \delta(t-t')$$

with, as causality suggests, $G(x) = 0$ for $x < 0$. The Green's function required is

$$G(x) = \frac{1}{\omega}\theta(t)\sin\omega t\,. \tag{3.28}$$

In terms of this function we have

$$\begin{aligned} u_1^\pm(t) &= a_s^\pm \sin\omega t + a_c^\pm \cos\omega t + 2\Delta\lambda_\pm \int_{-\infty}^{\infty} dt' G(t-t') \frac{df_1(t')}{dt'} \\ &= a_s^\pm \sin\omega t + a_c^\pm \cos\omega t + 2\Delta\lambda_\pm \frac{1}{\omega}\int_0^t dt' \frac{df_1(t')}{dt} \sin\left(\omega(t-t')\right) \end{aligned} \tag{3.29}$$

(The lower limit reflects the fact that f_1 is identically zero for negative argument.) Once we have $u_1^\pm$ we can get $v_1^\pm$ from Eq. (3.24a):

$$\begin{aligned} v_1^\pm &= \pm\frac{1}{\omega}\left(\frac{du_1^\pm}{dt} + K_\pm f_1\right) \\ &= \pm\left(a_s^\pm \cos\omega t - a_c^\pm \sin\omega t + \frac{K_\pm}{\omega} f_1(t)\right) \end{aligned} \tag{3.30}$$

At this point we can get $a_s^\pm$ and $a_c^\pm$ from the boundary conditions for $v_1^\pm(0)$ and $u_1^\pm(0)$, which determine $a_s^\pm = a_c^\pm = 0$. (It is worth noting that since these quantities are not proportional to f_1, the only way for them to be small is to be zero.) In turn, we have finally

$$\rho_1^\pm(t) = \frac{2\Delta\lambda_\pm}{\omega} \int_0^t \int_0^{t'} \sin\left(\omega\left(t' - t''\right)\right) \frac{df_1\left(t''\right)}{dt''} dt'' dt' \tag{3.31}$$

$$\sigma_1^\pm = \pm \frac{1}{\omega} K_\pm \int_0^t f_1\left(t'\right) dt' \tag{3.32}$$

The phase function $S_\pm(t)$ takes on a suggetive form if we use the identity $K_\pm = \pm\omega\lambda_\pm - \Delta(b_0 \pm \omega)$, in which case we can write

$$S_\pm(t) = \lambda_\pm \int_0^t \left(1 + f_1\left(t'\right)\right) dt' \mp \frac{2\Delta\lambda_\pm}{\omega} \int_0^t f_1\left(t'\right) dt' . \tag{3.33}$$

The first term integrates the material density.

An example: We take a linear variation, $f_1 = qt$, together with the condition that the beam is pure ν_μ at $t = 0$ (which translates into $\zeta_1(0) = -\sin\theta$ and $\zeta_2(0) = \cos\theta$). Then we have immediately

$$S_\pm(t) = \lambda_\pm \left(t + \frac{1}{2} q t^2\right) \mp \frac{2\Delta\lambda_\pm}{\omega} \frac{1}{2} q t^2 \tag{3.34a}$$

and

$$R_\pm(t) = R_\pm \left[1 + q \frac{2\Delta\lambda_\pm}{\omega^2} \left(t - \frac{1}{\omega} \sin\omega t\right)\right]. \tag{3.34b}$$

In Figs. 1 and 2 we plot some probabilities associated with the resulting amplitude for some representative values of the parameters.

Acknowledgements

We would like to thank the Aspen Center for Physics, where much of this work was done. PMF would also like to thank Dominique Schiff and the members of the LPTHE at Université de Paris-Sud for their generous hospitality. This work is supported in part by the U.S. Department of Energy under grant number DE-FG02 -97ER41027.

References

1. Super-Kamiokande Collaboration, Y. Fukuda *et al*., Phys. Lett. B, **433** (1998) 9; *ibid*., **436** (1998) 33; Phys. Rev. Lett. **81** (1998) 1562; *ibid*., **82** (1999) 1810; *ibid*., **82** (1999) 2430; E. Kearns, TAUP97, The 5th International Workshop on Topics in Astroparticle and Underground Physics, Nucl. Phys. Proc. Suppl. **70** (1999) 315; A. Habig for the Super-Kamiokande Collaboration, hep-ex/9903047; K. Scholberg for the Super-Kamiokande Collaboration, hep-ex/9905016, to appear in the Proceedings of the 8th International Workshop on Neutrino Telescopes (Venice, Italy, 1999); G. L. Fogli, E. Lisi, A. Marrone, and G. Scioscia, hep-ph/9904465, to appear in the Proceedings of *WIN '99*, 17th Annual Workshop on Weak Interactions and Neutrinos (Cape Town, South Africa, 1999); Kamiokande Collaboration, K. S. Hirata et al., Phys. Lett. **B280**, 146

(1992); Y. Fukuda *et al*., Phys. Lett. **B335** (1994) 237; IMB collaboration, R. Becker-Szendy *et al*., Nucl. Phys. B (Proc. Suppl.) **38** (1995) 331; Soudan-2 collaboration, W. W. M. Allison *et al*., Phys. Lett. **B391** (1997) 491; Kamiokande Collaboration, S. Hatekeyama *et al*., Phys. Rev. Lett. **81** (1998) 2016; MACRO Collaboration, M. Ambrosio *et al*., Phys. Lett. B **434** (1998) 451; CHOOZ Collaboration, M. Apollonio *et al*., Phys. Lett. **B420** (1998) 397.

2. J. N. Bahcall and M. H. Pinsonneault, Rev. Mod. Phys. **67** (1995) 781; J. N. Bahcall, S. Basu, and M. H. Pinsonneault, Phys. Lett. B **433** (1998) 1; J. N. Bahcall, P. I. Krastev, and A. Yu. Smirnov, Phys. Rev. **D58** (1998) 096016; B. T. Cleveland *et al*., Nucl. Phys. B (Proc. Suppl.) **38** (1995) 47; Kamiokande Collaboration, Y. Fukuda *et al*., Phys. Rev. Lett. **77** (1996) 1683; GALLEX Collaboration, W. Hampel *et al*., Phys. Lett. **B388** (1996) 384; SAGE Collaboration, J. N. Abdurashitov *et al*., Phys. Rev. Lett. **77** (1996) 4708; Liquid Scintillator Neutrino Detector (LSND) Collaboration, C. Athanassopoulos *et al*., Phys. Rev. Lett. **75** (1996) 2650; *ibid*., **77** (1996) 3082; Phys. Rev. **C58** (1998) 2489; Phys. Rev. Lett. **81** (1998)1774; G. L. Fogli, E. Lisi, A. Marrone, and G. Scioscia, Phys. Rev. **D59** (1999) 033001. See also G. L. Fogli, E. Lisi, and A. Marrone, Phys. Rev. **D57** (1998) 5893 and references therein.

3. S. P. Mikheyev and A. Yu. Smirnov, Sov. J. Nucl. Phys. **42**, 913 (1985).

4. L. Wolfenstein, Phys. Rev. **D17**, 2369 (1978).

5. The first paper to point out that a two-layer case is relevant to passage and conversion of neutrinos within the Earth is S. T. Petcov, Phys. Lett. B **434** (1998) 321. There is much subsequent work, including (and not necessarily in chronological order) M.V. Chizhov and S. T. Petcov, Phys. Rev. Lett. **83** (1999) 1096; M.V. Chizhov and S. T. Petcov, hep-ph/9903424; M.V. Chizhov, hep-ph/9909439; E. Kh. Akhmedov, Nucl. Phys. **B538**, 25 (1999); E. Kh. Akhmedov, A. Dighe, P. Lipari, and A. Yu. Smirnov, Nucl Phys. **B542**, 3 (1999); M. V. Chizhov, M. Maris, and S. T. Petcov, hep-ph/9810501; E. Kh. Akhmedov, hep-ph/9903302; E. Kh. Akhmedov, Pramana **54**, 47 (2000); E. Kh. Akhmedov and A. Yu. Smirnov, hep-ph/9910433; M. V. Chizhov and S. T. Petcov, hep-ph/0003110; I. Mocioiu and R. Shrock, Phys. Rev. **D62** (2000) 053017; P. M. Fishbane, Phys. Rev. **D62** (2000) 093009; M. Freund and T. Ohlsson, Mod. Phys. Lett. A **15** (2000) 867; T. Ohlsson and H. Snellman, Phys. Lett. B **474** (2000) 153 and **480** (2000) 419(E).

6. This is of course an approach that we are not the first to take. Among many possible citations, and beyond those of references 3 and 4, we single out S. T. Petcov, Phys. Lett. B **191** (1987) 299; T. Ohlsson and H. Snellman, J. Math. Phys. **41** (2000) 2768; P. Osland and T. T. Wu, Phys. Rev, **D62** (2000) 013008; H. Lehmann, P. Osland, and T. T. Wu, hep-ph/0006213

7. For other work on the “adiabatic” treatment of neutrino oscillation amplitudes, see ref. 3; S. Toshov, Phys. Lett. B **185** (1987) 177; S. T. Petcov and S. Toshov, Phys. Lett. B **187** (1987) 120; P. Langacker et al., Nucl Phys. **B282** (1987) 589. The approximation in

these citations is not, we believe, the one we have developed here. We shall discuss this matter elsewhere.

Acknowledgements

We would like to thank the Aspen Center for Physics, where much of this work was done. PMF would also like to thank Dominique Schiff and the members of the LPTHE at Université de Paris-Sud for their hospitality. This work is supported in part by the U.S. Department of Energy under grant number DE-FG02-97ER41027.

Figure Captions

Figure 1. Probability, as calculated in the adiabatic approximation described in the text, of ν_e as a function of time from production as a pure ν_μ at $t = 0$, in a medium with density factor $A_0(1 + qt)$, where $A_0 = 6 \times 10^9$ cm^{-1} = 10^{-13} eV corresponds to Earth-like density. We assume the primary mixing angle is $\theta = 0.7$ and that the difference of the square of the neutrino masses is 5×10^{-6} eV^2. The factor $\Delta = -7.9 \times 10^{-14}$ eV, a value for which the neutrino energy lies around the MSW resonance value, an energy of roughly 20 MeV. The horizontal exis is in units of 10^{14} eV^{-1}; note that for $q = -2 \times 10^{-15}$ eV, $qt = -0.2$ at $t = 10^{14}$ eV^{-1}.

Figure 2. Same as Fig. 1, but with the factor $\Delta = -7.5 \times 10^{-15}$ eV, a value for which the neutrino energy lies roughly ten times higher than that corresponding to the MSW resonance value.

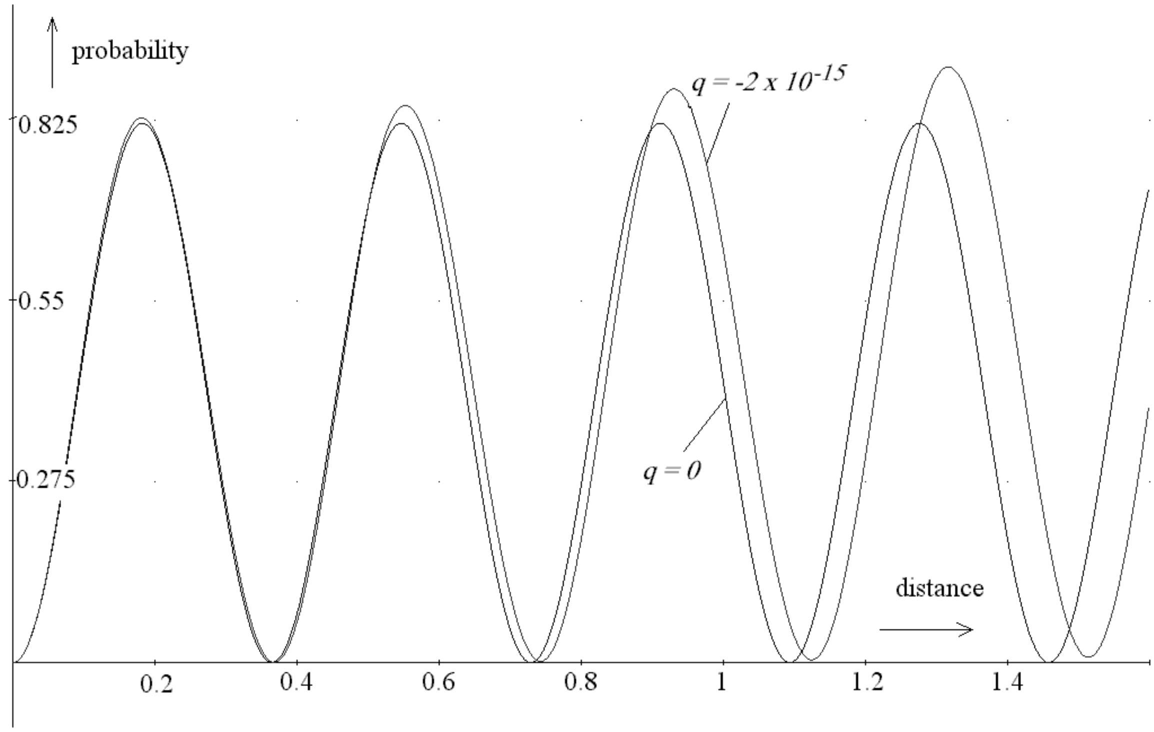

Figure 1

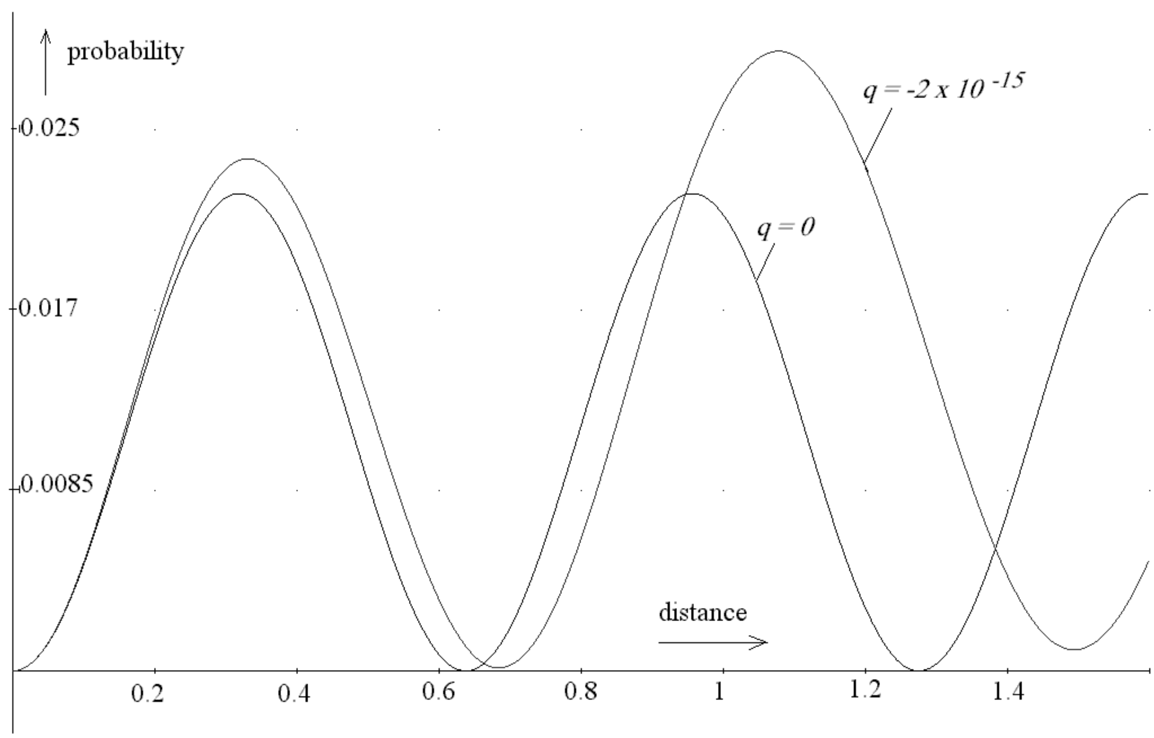

Figure 2